

Sun's log-concavity conjecture on the Catalan-Larcombe-French sequence

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Abstract. Let $\{P_n\}_{n \geq 0}$ denote the Catalan-Larcombe-French sequence, which naturally came up from the series expansion of the complete elliptic integral of the first kind. In this paper, we prove the strict log-concavity of the sequence $\{\sqrt[n]{P_n}\}_{n \geq 1}$, which was originally conjectured by Sun. We also obtain the strict log-concavity of the sequence $\{\sqrt[n]{V_n}\}_{n \geq 1}$, where $\{V_n\}_{n \geq 0}$ is the Fennessey-Larcombe-French sequence arising in the series expansion of the complete elliptic integral of the second kind.

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1 Introduction

The main objective of this paper is to prove the log-concavity of the sequence $\{\sqrt[n]{P_n}\}_{n \geq 1}$ and the sequence $\{\sqrt[n]{V_n}\}_{n \geq 1}$, where $\{P_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 0}$, known as the Catalan-Larcombe-French sequence and the Fennessey-Larcombe-French sequence respectively, are given by

$$(n+1)^2 P_{n+1} = 8(3n^2 + 3n + 1)P_n - 128n^2 P_{n-1}, \quad (1.1)$$

$$n(n+1)^2 V_{n+1} = 8n(3n^2 + 5n + 1)V_n - 128(n-1)(n+1)^2 V_{n-1}, \quad (1.2)$$

with the initial values $P_0 = V_0 = 1$ and $P_1 = V_1 = 8$. These two sequences came up naturally from the series expansions of the complete elliptic integrals. For more information on $\{P_n\}_{n \geq 0}$ and $\{V_n\}_{n \geq 0}$, see [1, 3, 4, 5, 6].

Let us first give an overview of some background. Recall that a sequence $\{a_n\}_{n \geq 0}$ of real numbers is said to be log-concave (resp. log-convex) if

$$a_n^2 \geq a_{n-1}a_{n+1} \quad (\text{resp. } a_n^2 \leq a_{n-1}a_{n+1})$$

for all $n \geq 1$, and it is strictly log-concave (resp. strictly log-convex) if the inequality is strict. Clearly, for some integer $N > 0$, the positive sequence $\{a_n\}_{n \geq N}$ is log-concave (resp. log-convex) if and only if the sequence $\{a_{n+1}/a_n\}_{n \geq N}$ is decreasing (resp. increasing).

The strict log-concavity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ was first conjectured by Sun [10], who also conjectured that the sequences $\{P_{n+1}/P_n\}_{n \geq 0}$ and $\{\sqrt[n]{P_n}\}_{n \geq 1}$ are strictly increasing. In

fact, motivated by Firoozbakht's conjecture on the strictly decreasing property of the sequence $\{\sqrt[n]{p_n}\}_{n \geq 1}$, where p_n is the n -th prime number, see [7, p.185], Sun [10] studied the monotonicity of many number theoretical and combinatorial sequences.

The strictly increasing property of the sequence $\{P_{n+1}/P_n\}_{n \geq 0}$ and $\{\sqrt[n]{P_n}\}_{n \geq 1}$ has been confirmed by Xia and Yao [12], and independently by Zhao [14]. We would like to point out that Zhao [14] only proved the log-convexity of $\{P_n\}_{n \geq 0}$, which also implies the monotonicity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ by using a result due to Wang and Zhu [11].

For a positive sequence $\{a_n\}_{n \geq 0}$ satisfying a three-term recurrence relation, Chen, Guo and Wang [2] obtained a useful criterion to determine the log-concavity of $\{\sqrt[n]{a_n}\}_{n \geq N}$ for some positive integer N . While, their criterion does not apply to the Catalan-Larcombe-French sequence, and Sun's conjecture on the log-concavity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ remains open. The first main result of this paper is as follows.

Theorem 1.1. *The sequence $\{\sqrt[n]{P_n}\}_{n \geq 1}$ is strictly log-concave, that is, for $n \geq 2$,*

$$\left(\sqrt[n]{P_n}\right)^2 > \sqrt[n-1]{P_{n-1}} \cdot \sqrt[n+1]{P_{n+1}}. \quad (1.3)$$

Since the Fennessey-Larcombe-French sequence $\{V_n\}_{n \geq 0}$ is closely related to the sequence $\{P_n\}_{n \geq 0}$, we are led to study the log-behavior of $\{\sqrt[n]{V_n}\}_{n \geq 1}$. The second main result of this paper is as follows.

Theorem 1.2. *The sequence $\{\sqrt[n]{V_n}\}_{n \geq 1}$ is strictly log-concave, that is, for $n \geq 2$,*

$$\left(\sqrt[n]{V_n}\right)^2 > \sqrt[n-1]{V_{n-1}} \cdot \sqrt[n+1]{V_{n+1}}. \quad (1.4)$$

This paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 by establishing a lower bound and an upper bound for the ratio P_n/P_{n-1} . The proof of Theorem 1.2 is similar to that of Theorem 1.1, which will be given in Section 3. Finally, we give a new proof of the monotonicity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ by using Theorem 1.1, and obtain the monotonicity of $\{\sqrt[n]{V_n}\}_{n \geq 1}$ by using Theorem 1.2.

2 Proof of Theorem 1.1

In this section we aim to prove Theorem 1.1. To this end, we first establish a lower bound and an upper bound for the ratio P_n/P_{n-1} that will lead to the strict log-concavity of the sequence $\{\sqrt[n]{P_n}\}_{n \geq 1}$.

Lemma 2.1. *For any integer $n \geq 1$, let*

$$f(n) = \frac{16(n-1)}{n}. \quad (2.1)$$

Then for $n \geq 5$, we have

$$f(n-1) < \frac{P_n}{P_{n-1}} < f(n). \quad (2.2)$$

Proof. For notational convenience, let $v(n) = P_n/P_{n-1}$. We first use induction on n to prove $v(n) > f(n-1)$ for $n \geq 5$. By the recurrence (1.1), we have

$$v(n+1) = \frac{8(3n^2 + 3n + 1)}{(n+1)^2} - \frac{128n^2}{(n+1)^2 v(n)}, \quad n \geq 1, \quad (2.3)$$

with the initial value $v(1) = 8$. Clearly, $v(5) = 2152/169 > 12 = f(4)$. Assume that $v(n) > f(n-1)$, and we proceed to prove that $v(n+1) > f(n)$ for $n \geq 5$. Observe that

$$\begin{aligned} v(n+1) - f(n) &= \frac{8(3n^2 + 3n + 1)}{(n+1)^2} - \frac{128n^2}{(n+1)^2 v(n)} - \frac{16(n-1)}{n} \\ &= \frac{8(n^3 + n^2 + 3n + 2)v(n) - 128n^3}{n(n+1)^2 v(n)}. \end{aligned}$$

By the induction hypothesis, we have $v(n) > f(n-1) > 0$ for $n \geq 5$, therefore

$$v(n+1) - f(n) > \frac{8(n^3 + n^2 + 3n + 2)f(n-1) - 128n^3}{n(n+1)^2 v(n)} = \frac{128(n^2 - 4n - 4)}{(n-1)n(n+1)^2 v(n)} > 0$$

for $n \geq 5$, since $n^2 - 4n - 4 = (n+1)(n-5) + 1 > 0$ for $n \geq 5$. This proves that $v(n) > f(n-1)$ for $n \geq 5$.

The inequality $v(n) < f(n)$ for $n \geq 5$ can be obtained in the same manner, and the detailed proof is omitted here. This completes the proof. \square

With the bounds given in Lemma 2.1 we are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Note that for $n \geq 2$, the inequality (1.3) can be rewritten as

$$\frac{\sqrt[n]{P_n}}{\sqrt[n-1]{P_{n-1}}} > \frac{\sqrt[n+1]{P_{n+1}}}{\sqrt[n]{P_n}},$$

or equivalently,

$$\left(\frac{P_n}{P_{n-1}} \right)^{n(n+1)} > P_n^2 \left(\frac{P_{n+1}}{P_n} \right)^{n(n-1)}. \quad (2.4)$$

By the recurrence (1.1), it is easy to verify that (2.4) holds for $2 \leq n \leq 6$. We proceed

to prove that (2.4) is true for $n \geq 7$. By Lemma 2.1 we have

$$\begin{aligned}
\left(\frac{P_n}{P_{n-1}}\right)^{n(n+1)} - P_n^2 \left(\frac{P_{n+1}}{P_n}\right)^{n(n-1)} &> (f(n-1))^{n(n+1)} - P_n^2 (f(n+1))^{n(n-1)} \\
&= \left(\frac{16(n-2)}{n-1}\right)^{n(n+1)} - P_n^2 \left(\frac{16n}{n+1}\right)^{n(n-1)} \\
&= 16^{n(n-1)} \left(16^{2n} \left(\frac{n-2}{n-1}\right)^{n(n+1)} - P_n^2 \left(\frac{n}{n+1}\right)^{n(n-1)}\right).
\end{aligned}$$

It suffices to prove that

$$16^{2n} \left(\frac{n-2}{n-1}\right)^{n(n+1)} - P_n^2 \left(\frac{n}{n+1}\right)^{n(n-1)} > 0,$$

or equivalently

$$\frac{P_n^2}{16^{2n}} < \left(\frac{(n-2)(n+1)}{(n-1)n}\right)^{n(n-1)} \left(\frac{n-2}{n-1}\right)^{2n}.$$

Thus we only need to show that, for $n \geq 7$,

$$\frac{P_n}{16^n} < \left(\frac{(n-2)(n+1)}{(n-1)n}\right)^{\frac{n(n-1)}{2}} \left(\frac{n-2}{n-1}\right)^n. \quad (2.5)$$

Let l_n denote the term on the left hand side, and r_n denote the term on the right hand side. We claim that

- (i) the sequence $\{l_n\}_{n \geq 5}$ is strictly decreasing, and
- (ii) the sequence $\{r_n\}_{n \geq 5}$ is strictly increasing.

By Lemma 2.1, we see that

$$0 < \frac{l_n}{l_{n-1}} = \frac{P_n/P_{n-1}}{16} < 1$$

for $n \geq 5$, which implies (i).

We proceed to prove (ii). Note that

$$r_n = \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}} \cdot \left(1 - \frac{1}{n-1}\right)^{n-1} \cdot \left(1 - \frac{1}{n-1}\right).$$

The increasing property of the sequence $\{r_n\}_{n \geq 5}$ immediately follows from the well-known fact that the sequence $\{(1 - \frac{1}{n})^n\}_{n \geq 1}$ is strictly increasing.

It is easy to verify that $l_7 < r_7$. Combining (i) and (ii), we get that $l_n < r_n$ for $n \geq 7$, namely (2.5) holds. This completes the proof. \square

3 Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2 in a similar way of that of Theorem 1.1. For this purpose, we need a lower bound and an upper bound for the ratio V_n/V_{n-1} . For integer $n \geq 2$, let

$$h(n) = \frac{16(n^3 - n^2 + 1)}{n^3 - n^2}, \quad (3.1)$$

which was introduced by Yang and Zhao [13] in their study of the log-concavity of the sequence $\{V_n\}_{n \geq 1}$. They [13] showed that $h(n)$ is a lower bound for the ratio V_n/V_{n-1} , precisely,

$$\frac{V_n}{V_{n-1}} > h(n), \quad (3.2)$$

for $n \geq 4$. We further show that $h(n-1)$ is an upper bound for the ratio V_n/V_{n-1} .

Lemma 3.1. *Let $h(n)$ be given by (3.1). Then for $n \geq 11$, we have*

$$\frac{V_n}{V_{n-1}} < h(n-1). \quad (3.3)$$

Proof. Let $g(n) = V_n/V_{n-1}$, by (1.2) it is clear that

$$g(n+1) = \frac{8(3n^2 + 5n + 1)}{(n+1)^2} - \frac{128(n-1)}{ng(n)}, \quad n \geq 1, \quad (3.4)$$

with $g(1) = 8$. The inequality (3.3) can be proved inductively based on the recurrence (3.4). The proof is similar to that of Lemma 2.1, and hence is omitted here. \square

We proceed to prove Theorem 1.2.

Proof of Theorem 1.2. Note that for $n \geq 2$, the inequality (1.4) can be rewritten as

$$\frac{\sqrt[n]{V_n}}{\sqrt[n-1]{V_{n-1}}} > \frac{\sqrt[n+1]{V_{n+1}}}{\sqrt[n]{V_n}},$$

or equivalently,

$$\left(\frac{V_n}{V_{n-1}}\right)^{n(n+1)} > V_n^2 \left(\frac{V_{n+1}}{V_n}\right)^{n(n-1)}. \quad (3.5)$$

By the recurrence (1.2), it is easy to verify that (3.5) holds for $2 \leq n \leq 9$. We proceed to prove that (3.5) is true for $n \geq 10$. By (3.2) we have

$$\frac{V_n}{V_{n-1}} > h(n),$$

for $n \geq 4$. By Lemma 3.1, we have

$$\frac{V_{n+1}}{V_n} < h(n),$$

for $n \geq 10$. Then for $n \geq 10$, it follows that

$$\begin{aligned} \left(\frac{V_n}{V_{n-1}}\right)^{n(n+1)} - V_n^2 \left(\frac{V_{n+1}}{V_n}\right)^{n(n-1)} &> (h(n))^{n(n+1)} - V_n^2 (h(n))^{n(n-1)} \\ &= (h(n))^{n(n-1)} ((h(n))^n + V_n) ((h(n))^n - V_n). \end{aligned}$$

Clearly, both $(h(n))^{n(n-1)}$ and $((h(n))^n + V_n)$ are positive for $n \geq 10$. Thus it suffices to show that for $n \geq 10$,

$$(h(n))^n - V_n > 0. \quad (3.6)$$

Note that the relation (24) in [5] gave an upper bound of V_n , that is, for $n \geq 1$,

$$V_n < (2n+1) \binom{2n}{n}^2. \quad (3.7)$$

Sasvári [9, Corollary 1] showed that for $n \geq 1$,

$$\binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}} e^{-\frac{1}{8n} + \frac{1}{192n^3}}. \quad (3.8)$$

It is clear that for $n \geq 1$,

$$0 < e^{-\frac{1}{8n} + \frac{1}{192n^3}} < 1.$$

Then it follows from (3.8) that for $n \geq 1$

$$\binom{2n}{n} < \frac{4^n}{\sqrt{\pi n}}. \quad (3.9)$$

Combining (3.7) and (3.9), for $n \geq 1$ we have

$$V_n < \frac{2n+1}{\pi n} 16^n < 16^n. \quad (3.10)$$

Note that for $n \geq 2$

$$h(n) = \frac{16(n^3 - n^2 + 1)}{n^3 - n^2} > 16. \quad (3.11)$$

Thus by (3.10) and (3.11), we obtain (3.6). This completes the proof. \square

4 The monotonicity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ and $\{\sqrt[n]{V_n}\}_{n \geq 1}$

In this section, we aim to derive the monotonicity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$ from the log-concavity of $\{\sqrt[n]{P_n}\}_{n \geq 1}$, and to derive the monotonicity of $\{\sqrt[n]{V_n}\}_{n \geq 1}$ from the log-concavity of $\{\sqrt[n]{V_n}\}_{n \geq 1}$. The main result of this section is as follows.

Proposition 4.1. *Both $\{\sqrt[n]{P_n}\}_{n \geq 1}$ and $\{\sqrt[n]{V_n}\}_{n \geq 1}$ are strictly increasing.*

Proof. Recall that for a real sequence $\{a_n\}_{n \geq 0}$ with positive numbers, it was shown that

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{a_n},$$

and

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n},$$

see Rudin [8, §3.37]. These two inequalities imply a well-known criterion, that is, if $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = c$, then $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = c$, where c is a real number.

Note that it was proved in [5, Eq. (30)] that

$$\lim_{n \rightarrow \infty} \frac{P_n}{P_{n-1}} = \lim_{n \rightarrow \infty} \frac{V_n}{V_{n-1}} = 16.$$

Then it follows that

$$\lim_{n \rightarrow \infty} \sqrt[n]{P_n} = \lim_{n \rightarrow \infty} \sqrt[n]{V_n} = 16.$$

Consequently,

$$\lim_{n \rightarrow \infty} \sqrt[n+1]{P_{n+1}} / \sqrt[n]{P_n} = \lim_{n \rightarrow \infty} \sqrt[n+1]{V_{n+1}} / \sqrt[n]{V_n} = 1.$$

By Theorems 1.1 and 1.2, we see that both $\{\sqrt[n+1]{P_{n+1}} / \sqrt[n]{P_n}\}_{n \geq 1}$ and $\{\sqrt[n+1]{V_{n+1}} / \sqrt[n]{V_n}\}_{n \geq 1}$ are strictly decreasing. Thus for all $n \geq 1$, we have

$$\sqrt[n+1]{P_{n+1}} / \sqrt[n]{P_n} > 1, \quad \text{and} \quad \sqrt[n+1]{V_{n+1}} / \sqrt[n]{V_n} > 1.$$

This completes the proof. \square

Remark. By employing a criterion due to Wang and Zhu [11, Theorem 2.1], Yang and Zhao [13] showed that the sequence $\{\sqrt[n]{V_{n+1}}\}_{n \geq 1}$ is strictly decreasing.

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